

OPTIMAL CONSTANTS FOR A MIXED LITTLEWOOD TYPE INEQUALITY

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ABSTRACT. For $p \in [2, \infty]$ a mixed Littlewood-type inequality asserts that there is a constant $C_{(m),p} \geq 1$ such that

$$\left(\sum_{i_1=1}^{\infty} \left(\sum_{i_2, \dots, i_m=1}^{\infty} |T(e_{i_1}, \dots, e_{i_m})|^2 \right)^{\frac{1}{2} \frac{p-1}{p-1}} \right)^{\frac{p-1}{p}} \leq C_{(m),p} \|T\|$$

for all continuous real-valued m -linear forms on $\ell_p \times c_0 \times \dots \times c_0$ (when $p = \infty$, ℓ_p is replaced by c_0). We prove that for $p > 2.18006$ the optimal constants $C_{(m),p}$ are $\left(2^{\frac{1}{2} - \frac{1}{p}}\right)^{m-1}$. When $p = \infty$, we recover the best constants of the mixed (ℓ_1, ℓ_2) -Littlewood inequality.

1. INTRODUCTION

The Hardy–Littlewood inequality ([17], 1934) is a continuation of famous works of Littlewood ([18], 1930) and Bohnenblust and Hille ([9], 1931) and can be stated as follows:

- [17, Theorems 2 and 4] If $p, q \geq 2$ are such that

$$\frac{1}{2} < \frac{1}{p} + \frac{1}{q} < 1$$

then there is a constant $C_{p,q} \geq 1$ such that

$$(1) \quad \left(\sum_{j,k=1}^{\infty} |A(e_j, e_k)|^{\frac{pq}{pq-p-q}} \right)^{\frac{pq-q-p}{pq}} \leq C_{p,q} \|A\|$$

for all continuous bilinear forms $A : \ell_p \times \ell_q \rightarrow \mathbb{R}$ (or \mathbb{C}). Moreover the exponent $\frac{pq}{pq-p-q}$ is optimal.

- [17, Theorems 1 and 4] If $p, q \geq 2$ are such that

$$\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$$

then there is a constant $C_{p,q} \geq 1$ such that

$$(2) \quad \left(\sum_{j,k=1}^{\infty} |A(e_j, e_k)|^{\frac{4pq}{3pq-2p-2q}} \right)^{\frac{3pq-2p-2q}{4pq}} \leq C_{p,q} \|A\|$$

for all continuous bilinear forms $A : \ell_p \times \ell_q \rightarrow \mathbb{R}$ (or \mathbb{C}). Moreover the exponent $\frac{4pq}{3pq-2p-2q}$ is optimal.

Above and henceforth, as usual in this field, when p and/or q is infinity, we consider c_0 instead of ℓ_p and/or ℓ_q .

As mentioned in [20, Theorem 1] an unified version of the above two results of Hardy and Littlewood asserts that there is a constant $C_{p,q} \geq 1$ such that

$$(3) \quad \left(\sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} |A(e_j, e_k)|^2 \right)^{\frac{\lambda}{2}} \right)^{\frac{1}{\lambda}} \leq C_{p,q} \|A\|$$

with $\lambda = \frac{pq}{pq-p-q}$, for all continuous bilinear forms $A : \ell_p \times \ell_q \rightarrow \mathbb{R}$ (in fact, in [20, Theorem 1] just the complex case is considered, but for a general approach including the real case we refer to [11]; moreover the exponents are optimal). The recent years witnessed an increasing interest in the study of summability of multilinear operators

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(see, for instance, [10, 23, 24]) and in estimating constants of the multilinear and polynomial Hardy–Littlewood and related inequalities (see [2, 3, 4, 6, 14, 15, 26]). Perhaps the main motivations are potential applications (see, for instance, [19] for applications of the real-valued case of the estimates of the Bohnenblust–Hille inequality and [7, 12] for applications of the complex-valued case).

One of the most for reaching generalizations of the Hardy–Littlewood inequality is the following theorem (see also [25]):

Theorem 1.1. (See Albuquerque, Araujo, Núñez, Pellegrino and Rueda [1]) *Let $m \geq 2$ be a positive integer, $1 \leq k \leq m$ and $n_1, \dots, n_k \geq 1$ be positive integers such that $n_1 + \dots + n_k = m$. If $q_1, \dots, q_k \in \left[\frac{1}{1 - \left(\frac{1}{p_1} + \dots + \frac{1}{p_m} \right)}, 2 \right]$ and $0 \leq \frac{1}{p_1} + \dots + \frac{1}{p_m} \leq \frac{1}{2}$, then the following assertions are equivalent:*

(a) *There is a constant $C_k = C(k, p_1, \dots, p_m, q_1, \dots, q_k)$ such that*

$$\left(\sum_{i_1=1}^{\infty} \left(\dots \left(\sum_{i_k=1}^{\infty} |T(e_{i_1}^{n_1}, \dots, e_{i_k}^{n_k})|^{q_k} \right)^{\frac{q_k-1}{q_k}} \dots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq C_k \|T\|$$

for all continuous m -linear forms $T : \ell_{p_1} \times \dots \times \ell_{p_m} \rightarrow \mathbb{R}$.

(b) *The numbers q_1, \dots, q_k satisfy*

$$\frac{1}{q_1} + \dots + \frac{1}{q_k} \leq \frac{k+1}{2} - \left(\frac{1}{p_1} + \dots + \frac{1}{p_m} \right).$$

Above, the notation $e_j^{n_j}$ represents the n_j -tuple (e_j, \dots, e_j) . The optimal constants of the previous inequalities are essentially unknown. Recent works have shown that in general these constants have a sublinear growth (see [5, 6, 7], and references therein). One of the few cases in which the optimal constants are known for all m is the case of mixed (ℓ_1, ℓ_2) -Littlewood inequality (see [21]):

- The optimal constants $C_{(m),\infty}$ satisfying

$$(4) \quad \sum_{i_1=1}^{\infty} \left(\sum_{i_2, \dots, i_m=1}^{\infty} |T(e_{i_1}, \dots, e_{i_m})|^2 \right)^{\frac{1}{2}} \leq C_{(m),\infty} \|T\|$$

for all continuous real m -linear forms $T : c_0 \times \dots \times c_0 \rightarrow \mathbb{R}$ are $2^{\frac{m-1}{2}}$.

From now on $p_0 \approx 1.84742$ is the unique real number satisfying

$$(5) \quad \Gamma \left(\frac{p_0 + 1}{2} \right) = \frac{\sqrt{\pi}}{2}.$$

Our main result provides the optimal constants of a Hardy–Littlewood-type inequality that encompasses (4); as far as we know this is the first time in which a Hardy–Littlewood type inequality (except for the case of mixed (ℓ_1, ℓ_2) -Littlewood inequality) is proved to have optimal constants with exponential growth:

Theorem 1.2. *Let $m \geq 2$ be a positive integer and $p \geq \frac{p_0}{p_0-1} \approx 2.18006$. The optimal constant $C_{(m),p}$ such that*

$$(6) \quad \left(\sum_{i_1=1}^{\infty} \left(\sum_{i_2, \dots, i_m=1}^{\infty} |T(e_{i_1}, \dots, e_{i_m})|^2 \right)^{\frac{1}{2} \frac{p}{p-1}} \right)^{\frac{p-1}{p}} \leq C_{(m),p} \|T\|,$$

for all continuous m -linear forms $T : \ell_p \times c_0 \times \dots \times c_0 \rightarrow \mathbb{R}$ is $\left(2^{\frac{1}{2} - \frac{1}{p}} \right)^{m-1}$.

Note that the above Hardy–Littlewood type inequality holds for $p \geq 2$ (see Theorem 1.1). When $p = 2$ it is simple to prove that the optimal constants are $C_{(m),p} = 1$. As a consequence of the arguments of our proof of Theorem 1.2 we remark that for $2 < p < \frac{p_0}{p_0-1}$ the optimal constants still have exponential growth; so an eventual decrease on the order of the growth when $p \rightarrow 2$ does not happen. Moreover, for $2 < p < \frac{p_0}{p_0-1} \approx 2.18006$, the difference between the bases in the exponential upper and lower estimates of $C_{(m),p}$ is not bigger than $4 \cdot 10^{-4}$ (see the figures 1 and 2).

In the final section we also provide upper and lower estimates for the sharp constants $C_{p,\infty}$ of the real case of (2), showing that

$$2^{\frac{1}{2}-\frac{1}{p}} \leq C_{p,\infty} \leq 2^{\frac{1}{2}-\frac{1}{2p}}$$

for all $p \geq \frac{p_0}{p_0-1} \approx 2.18006$. This result recovers, in particular, the optimality of the constant $\sqrt{2}$ of the real case of the Littlewood's $4/3$ inequality obtained in [15].

2. THE PROOF OF THEOREM 1.2

The Khinchine inequality (see [13]) asserts that, for any $0 < q < \infty$, there are positive constants A_q, B_q such that regardless of the scalar sequence $(a_j)_{j=1}^n$ we have

$$A_q \left(\sum_{j=1}^n |a_j|^2 \right)^{\frac{1}{2}} \leq \left(\int_0^1 \left| \sum_{j=1}^n a_j r_j(t) \right|^q dt \right)^{\frac{1}{q}} \leq B_q \left(\sum_{j=1}^n |a_j|^2 \right)^{\frac{1}{2}},$$

where r_j are the Rademacher functions. For real scalars, U. Haagerup [16] proved that if p_0 is the number defined in (5) then

$$A_q = \sqrt{2} \left(\frac{\Gamma(\frac{q+1}{2})}{\sqrt{\pi}} \right)^{\frac{1}{q}}, \quad \text{for } 1.84742 \approx p_0 < q < 2$$

and

$$A_q = 2^{\frac{1}{2}-\frac{1}{q}}, \quad \text{for } 1 \leq q \leq p_0 \approx 1.84742.$$

Let $T : \ell_p \times c_0 \times \cdots \times c_0 \rightarrow \mathbb{R}$ be a continuous m -linear form. By the Khinchine inequality for multiple sums (see [22]) we know that

$$\begin{aligned} & \left(\sum_{i_1=1}^{\infty} \left(\sum_{i_2, \dots, i_m=1}^{\infty} |T(e_{i_1}, \dots, e_{i_m})|^2 \right)^{\frac{1}{2} \frac{p-1}{p}} \right)^{\frac{p-1}{p}} \\ & \leq (A_{\frac{p}{p-1}}^{-1})^{m-1} \left(\sum_{i_1=1}^{\infty} \int_{[0,1]^{m-1}} \left| \sum_{i_2, \dots, i_m}^{\infty} r_{i_2}(t_2) \cdots r_{i_m}(t_m) T(e_{i_1}, \dots, e_{i_m}) \right|^{\frac{p}{p-1}} dt_2 \cdots dt_m \right)^{\frac{p-1}{p}} \\ & = (A_{\frac{p}{p-1}}^{-1})^{m-1} \left(\int_{[0,1]^{m-1}} \sum_{i_1=1}^{\infty} \left| T \left(e_{i_1}, \sum_{i_2=1}^{\infty} r_{i_2}(t_2) e_{i_2}, \dots, \sum_{i_m=1}^{\infty} r_{i_m}(t_m) e_{i_m} \right) \right|^{\frac{p}{p-1}} dt_2 \cdots dt_m \right)^{\frac{p-1}{p}} \\ & \leq (A_{\frac{p}{p-1}}^{-1})^{m-1} \left(\int_{[0,1]^{m-1}} \left\| T \left(\cdot, \sum_{i_2=1}^{\infty} r_{i_2}(t_2) e_{i_2}, \dots, \sum_{i_m=1}^{\infty} r_{i_m}(t_m) e_{i_m} \right) \right\|^{\frac{p}{p-1}} dt_2 \cdots dt_m \right)^{\frac{p-1}{p}} \\ & \leq (A_{\frac{p}{p-1}}^{-1})^{m-1} \sup_{t_2, \dots, t_m \in [0,1]} \left\| T \left(\cdot, \sum_{i_2=1}^{\infty} r_{i_2}(t_2) e_{i_2}, \dots, \sum_{i_m=1}^{\infty} r_{i_m}(t_m) e_{i_m} \right) \right\| \\ & \leq (A_{\frac{p}{p-1}}^{-1})^{m-1} \|T\| = (2^{\frac{1}{2}-\frac{1}{p}})^{m-1} \|T\| \end{aligned}$$

whenever $p \geq \frac{p_0}{p_0-1} \approx 2.18006$. Now let us show that $(2^{\frac{1}{2}-\frac{1}{p}})^{m-1}$ is the best possible constant. Let $T_2 : \ell_p^2 \times \ell_\infty^2 \rightarrow \mathbb{R}$ and $T_2^{x_2} : \ell_p^2 \rightarrow \mathbb{R}$ be given by

$$(7) \quad T_2(x_1, x_2) = (x_2^1 + x_2^2) x_1^1 + (x_2^1 - x_2^2) x_1^2,$$

and

$$T_2^{x_2}(x_1) = T_2(x_1, x_2),$$

for each $x_2 \in \ell_\infty^2$. Observe that

$$(8) \quad \|T_2\| = \sup \left\{ \|T_2^{x_2}\| : \|x_2\|_{\ell_\infty^2} = 1 \right\}.$$

Let us estimate (8). Since $(\ell_p)^* = \ell_{\frac{p}{p-1}}$, we have

$$\begin{aligned}
 (9) \quad \|T_2\| &= \sup \left\{ \|T_2^{x_2}\| : \|x_2\|_{\ell_\infty^2} = 1 \right\} \\
 &= \sup \left\{ \sup_{x_1 \in B_{\ell_p^2}} |T_2^{x_2}(x_1)| : \|x_2\|_{\ell_\infty^2} = 1 \right\} \\
 &= \sup \left\{ \sup_{x_1 \in B_{\ell_p^2}} |(x_2^1 + x_2^2)x_1^1 + (x_2^1 - x_2^2)x_1^2| : \|x_2\|_{\ell_\infty^2} = 1 \right\} \\
 &= \sup \left\{ \|(x_2^1 + x_2^2, x_2^1 - x_2^2, 0, 0, \dots)\|_{\ell_{\frac{p}{p-1}}} : \|x_2\|_{\ell_\infty^2} = 1 \right\} \\
 &= \sup \left\{ \left(|1 + x|^{\frac{p}{p-1}} + |1 - x|^{\frac{p}{p-1}} \right)^{\frac{1}{p-1}} : x \in [-1, 1] \right\} = 2.
 \end{aligned}$$

In order to verify the last equality, note that since

$$\sup \left\{ \left(|1 + x|^1 + |1 - x|^1 \right)^1 ; x \in [-1, 1] \right\} = 2,$$

by the norm inclusion $\ell_1 \subset \ell_{\frac{p}{p-1}}$ for $p \in [2, \infty)$, we have $\|\cdot\|_{\ell_{\frac{p}{p-1}}} \leq \|\cdot\|_{\ell_1}$. Therefore, for $p \in [2, \infty)$ we have

$$\sup \left\{ \left(|1 + x|^{\frac{p}{p-1}} + |1 - x|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} ; x \in [-1, 1] \right\} \leq \sup \left\{ \left(|1 + x|^1 + |1 - x|^1 \right)^1 ; x \in [-1, 1] \right\} = 2.$$

On the other hand, it is obvious that

$$\sup \left\{ \left(|1 + x|^{\frac{p}{p-1}} + |1 - x|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} ; x \in [-1, 1] \right\} \geq \left(|1 + 1|^{\frac{p}{p-1}} + |1 - 1|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} = 2.$$

In order to show that $\left(2^{\frac{1}{2} - \frac{1}{p}}\right)^{m-1}$ is the best possible constant satisfying (6), let T_2 be as in (7) and define for all $m \geq 3$ the m -linear operator $T_m : \ell_p^{2^{m-1}} \times \ell_\infty^{2^{m-1}} \times \dots \times \ell_\infty^{2^{m-1}} \rightarrow \mathbb{R}$ by

$$\begin{aligned}
 T_m(x_1, \dots, x_m) &= (x_m^1 + x_m^2)T_{m-1}(x_1, \dots, x_{m-1}) \\
 &\quad + (x_m^1 - x_m^2)T_{m-1}(S_p^{2^{m-2}}(x_1), S_0^{2^{m-2}}(x_2), S_0^{2^{m-3}}(x_3), \dots, S_0^2(x_{m-1})),
 \end{aligned}$$

where $x_1 \in \ell_p^{2^{m-1}}$, $x_k \in \ell_\infty^{2^{m-1}}$ for all $k = 2, \dots, m$, and $S_p : \ell_p \rightarrow \ell_p$ and $S_0 : c_0 \rightarrow c_0$ are the backward shifts. By induction on $m \geq 2$ we shall show that

$$\|T_m\| = 2^{m-1}.$$

The case $m = 2$ is already done in (9). Let us suppose that $\|T_{m-1}\| = 2^{(m-1)-1}$. Therefore,

$$\begin{aligned}
 |T_m(x_1, \dots, x_m)| &\leq |x_m^1 + x_m^2| |T_{m-1}(x_1, \dots, x_{m-1})| \\
 &\quad + |x_m^1 - x_m^2| |T_{m-1}(S_p^{2^{m-2}}(x_1), S_0^{2^{m-2}}(x_2), S_0^{2^{m-3}}(x_3), \dots, S_0^2(x_{m-1}))| \\
 &\leq 2^{m-2} [|x_m^1 + x_m^2| \|x_1\|_{\ell_p^{2^{m-1}}} \cdots \|x_{m-1}\|_{\ell_\infty^{2^{m-1}}} \\
 &\quad + |x_m^1 - x_m^2| \|S_p^{2^{m-2}}(x_1)\|_{\ell_p^{2^{m-1}}} \|S_0^{2^{m-2}}(x_2)\|_{\ell_\infty^{2^{m-1}}} \|S_0^{2^{m-3}}(x_3)\|_{\ell_\infty^{2^{m-1}}} \cdots \|S_0^2(x_{m-1})\|_{\ell_\infty^{2^{m-1}}}] \\
 &\leq 2^{m-2} [|x_m^1 + x_m^2| + |x_m^1 - x_m^2|] \|x_1\|_{\ell_p^{2^{m-1}}} \cdots \|x_{m-1}\|_{\ell_\infty^{2^{m-1}}} \\
 &= 2^{m-1} \|x_1\|_{\ell_p^{2^{m-1}}} \cdots \|x_{m-1}\|_{\ell_\infty^{2^{m-1}}} \max\{|x_m^1|, |x_m^2|\} \\
 &\leq 2^{m-1} \|x_1\|_{\ell_p^{2^{m-1}}} \cdots \|x_m\|_{\ell_\infty^{2^{m-1}}}.
 \end{aligned}$$

We thus have $\|T_m\| \leq 2^{m-1}$. Now consider $a_m = e_1 + e_2$ and note that

$$\begin{aligned}
 \|T_m\| &\geq \sup \left\{ |T_m(x_1, \dots, x_{m-1}, a_m)| : x_1 \in B_{\ell_p^{2^{m-1}}}, x_2 \in B_{\ell_\infty^{2^{m-1}}}, \dots, x_{m-1} \in B_{\ell_\infty^{2^{m-1}}} \right\} \\
 &= 2 \|T_{m-1}\| = 2^{m-1}
 \end{aligned}$$

and hence $\|T_m\| = 2^{m-1}$.

Since

$$\frac{\left(\sum_{i_1} \left(\sum_{i_2, \dots, i_m} |T_m(e_{i_1}, \dots, e_{i_m})|^2\right)^{\frac{1}{2} \frac{p}{p-1}}\right)^{\frac{p-1}{p}}}{\|T_m\|} = \left(2^{\frac{1}{2} - \frac{1}{p}}\right)^{m-1},$$

the proof is done.

3. FINAL REMARKS

The same argument used in the proof of Theorem 1.2 shows that for $2 < p < \frac{p_0}{p_0-1} \approx 2.18006$ the optimal constants also have exponential growth; curiously, for $p = 2$ the situation is quite different and the optimal constants are 1. In fact, note that the second part of the proof (the optimality proof) holds for all $p \geq 2$. Moreover, the first part of the proof gives us the estimate $C_{(m),p} \leq \left(A_{\frac{p}{p-1}}^{-1}\right)^{m-1}$. We thus have, for $2 \leq p < \frac{p_0}{p_0-1} \approx 2.18006$, the following inequalities

$$\left(2^{\frac{1}{2} - \frac{1}{p}}\right)^{m-1} \leq C_{(m),p} \leq \left(\frac{1}{\sqrt{2}} \left(\frac{\Gamma\left(\frac{2p-1}{2p-2}\right)}{\sqrt{\pi}}\right)^{\frac{1-p}{p}}\right)^{m-1}.$$

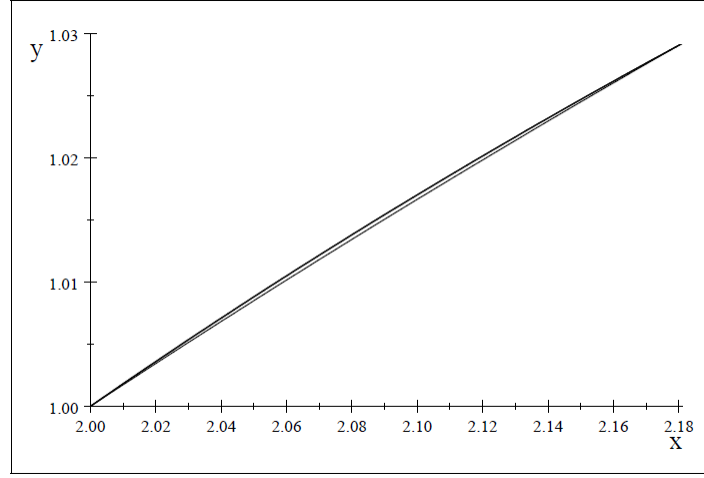


FIGURE 1. Plots of the functions $A_{\frac{x}{x-1}}^{-1}$ and $2^{\frac{1}{2} - \frac{1}{x}}$, for $x \in [2, \frac{p_0}{p_0-1}]$

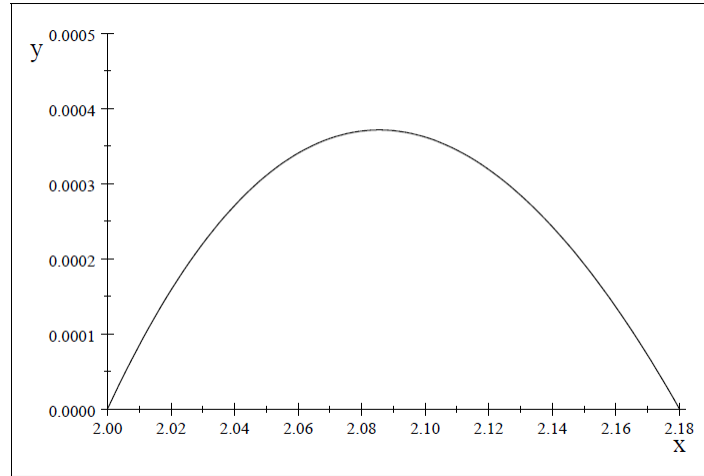


FIGURE 2. Plot of the function $\left(A_{\frac{x}{x-1}}^{-1} - 2^{\frac{1}{2} - \frac{1}{x}}\right)$, for $x \in [2, \frac{p_0}{p_0-1}]$

For $p \geq 2$, we know that

$$(10) \quad \left(\sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} |A(e_j, e_k)|^{\lambda} \right)^{\frac{1}{\lambda} 2} \right)^{\frac{1}{2}} \leq \sqrt{2} \|A\|$$

with $\lambda = \frac{p}{p-1}$, for all continuous bilinear forms $A : \ell_p \times c_0 \rightarrow \mathbb{R}$ (see, for instance, [2, Theorem 1.2 and Remark 5.1]). By interpolating (10) and the result of Theorem 1.2 for $m = 2$ in the sense of [2] or using the Hölder inequality for mixed sums ([8]) we obtain, for $p \geq \frac{p_0}{p_0-1} \approx 2.18006$,

$$\left(\sum_{j,k=1}^{\infty} |A(e_j, e_k)|^{\frac{4p}{3p-2}} \right)^{\frac{3p-2}{4p}} \leq \left(\sqrt{2} \|A\| \right)^{1/2} \left(2^{\frac{p-2}{2p}} \|A\| \right)^{1/2} = 2^{\frac{1}{2} - \frac{1}{2p}} \|A\|.$$

Using the approach of the previous section we obtain the lower estimate

$$C_{p,\infty} \geq \frac{\left(\sum_{j,k=1}^2 |T_2(e_j, e_k)|^{\frac{4p}{3p-2}} \right)^{\frac{3p-2}{4p}}}{\|T_2\|} = \frac{4^{\frac{3p-2}{4p}}}{2} = 2^{\frac{1}{2} - \frac{1}{p}}$$

and thus

$$2^{\frac{1}{2} - \frac{1}{p}} \leq C_{p,\infty} \leq 2^{\frac{1}{2} - \frac{1}{2p}}.$$

When $p = \infty$ we recover the well known optimal estimate of the famous Littlewood's 4/3 that can be found in [15].

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